Asymptotic entanglement in 1D quantum walks with a time-dependent coined

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Abstract

Discrete-time quantum walk evolve by a unitary operator which involves two operators a conditional shift in position space and a coin operator. This operator entangles the coin and position degrees of freedom of the walker. In this paper, we investigate the asymptotic behavior of the coin position entanglement (CPE) for an inhomogeneous quantum walk which determined by two orthogonal matrices in one-dimensional lattice. Free parameters of coin operator together provide many conditions under which a measurement perform on the coin state yield the value of entanglement on the resulting position quantum state. We study the problem analytically for all values that two free parameters of coin operator can take and the conditions under which entanglement becomes maximal are sought.

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1 Introduction

Among classical algorithms, many are based on classical random walk. Markov chain simulation, which has emerged as a powerful algorithmic tool [1] is one such example. Like classical random walk, the quantum version of it has also become an important constituent of quantum algorithms and computation. Though quantum random walk was first introduced by Aharonov et al. [2], but unlike its classical counterpart, the evolution of the quantum version is unitary, reversible and has no randomness associated with it during the evolution. Therefore, keeping away the term 'random', quantum walk has been the preferred usage. In the classical random walk the particle moves in the position space with a certain probability, whereas the quantum walk, which involves a superposition of states, moves by exploring multiple possible paths simultaneously with the amplitudes corresponding to different paths interfering. This properties makes the variance of the quantum walk on a line to grow quadratically with the number of steps (time), compared to the linear growth for the classical random walk [3, 4]. Quantum walk is studied in two standard forms: continuous-time quantum walk and discrete-time quantum walk that was introduced by Farhi and Gutmann [5] and Watrous [6], respectively. In the continuous-time quantum walk, one can directly define the walk on the position Hilbert space [5], but in the discrete-time quantum walk, in addition to it is necessary to introduce a coin Hilbert space to define the direction in which the particle has to move [3]. Due to the coin degree of freedom, the discrete-time variant is shown to be more powerful than the other in some contexts [7]. Childs describe a precise correspondence between continuous- and discrete-time quantum walks on arbitrary graphs [8]. Both these types of quantum walk have been widely used in algorithms for a diverse of problems. See for example [9, 10, 11, 12, 13, 14]. Beyond applications in quantum algorithms, quantum walk is emerging as a potential tool to understand various phenomenon in physical systems and has been employed to demonstrate coherent control over quantum many body systems. See for example [15, 16, 17, 18]. Some experimental

progress on the implementation of quantum walk has been reported [19, 20, 21, 22, 23, 24]. Entanglement is one of the attractive properties of quantum that does not appear in classic and is a very useful resource to perform various quantum tasks, a recent review by Horodecki et al. [25] discuss many of these aspects. Unfortunately, physical limitations and noise effects change the amount of entanglement and restrict it's efficiency. One way of circumventing this problem would be to generate entanglement. Quantum walk is one such process in which the conditional shift operator, which governs the itinerary of the quantum walker, induces entanglement between the degree of freedom of the coin and the spatial degree of freedom of the walker. This entanglement fluctuates with each step and eventually settles down to an asymptotic value that depends on the initial state of the quantum walk and on any bias in the quantum coin operator. Entanglement is a basic resource in quantum algorithms, and further work is required in order to fully understand its properties in the QW. While, entanglement between coin and position of the walker, as a type of quantum correlation, distort the distribution of the quantum walk and give it peaks and troughs, especially at the ends of the top hat [31, 32], it is necessary to obtain linear spreading and mixing times [33] and can be use to gauge the impact of the added randomness and decoherence [30]. In addition, it can be exploited for quantum information and communication purposes [26, 27, 28, 29]. Entanglement has also given us new insights for understanding many physical phenomena including super-radiance [34], superconductivity [35], disordered systems [36] and emerging of classicality [37]. In particular, understanding the role of entanglement in the existing methods of simulations of quantum spin systems allowed for significant improvement of the methods, as well as understanding their limitations [38]. So, studying entanglement during the quantum walk process will be useful from a quantum information theory perspective and also contribute to further investigation of the practical applications of the quantum walk. Carneiro et al. [41] studied the long-time asymptotic coin-position entanglemant of quantum walks on various graphs (Z , Z_2 , triangular lattices, cycles). Venegas-Andraca and Bose [42] also investigated

generation of entanglement between two walkers. Using Fourier analysis techniques, Abal et al. [43] analytically computed asymptotic coin position entanglement of the Hadamard walk on one dimension for both localized and non-localized initial conditions and in the same way Annabestani et al. [44] studied asymptotic entanglement in a two-dimensional quantum walk. Chandrashekar et al. [45] consider a multipartite quantum walk on a one-dimensional lattice and studeid the evolution of spatial entanglement, entanglement between different lattice points.

Most of the original work considered homogeneous walks, which the amplitude for moving does not depend on position. The idea of looking at inhomogeneous walks is not new, the recognition that it is natural to allow coins to be position dependent may be found in [46, 47, 48, 49, 50, 55]. To probe the possible long time behaviours and quantify the entanglement between the coin and position of such walks is our motivation for this work. In this paper, we investigate the asymptotic behavior of the CPE for an inhomogeneous QW which is determined by two orthogonal matrices in one-dimensional lattice. Our inhomogeneous QW is a kind of the generalized model defined in [55, 56] which is based on the idea of the Aubry-André model [57]. The limit distribution of the this QW has the probability density given by Dirac delta function which is called a localization for the inhomogeneous QW (for more details see Ref.[55]). We show how the entanglement approaches asymptotic values depending on the choice of initial state and coin bias with study the problem analytically for all values of θ_0 and θ_1 for large number of QW steps. We find that CPE is different for odd or even positions and when on the coin parameters took the precise value $\frac{\pi}{2}$ walk alter to a bounded motion.

The plan of the paper is as follows: in section 2 the model for discrete-time quantum walk is presented in detail. Section 3 defines the entanglement and provides the required formalism in the Fourier space leading to the long-time reduced density operator for arbitrary coin operations and initial states. Our inhomogeneous walk with two-period has defined in section 4 and the asymptotic CPE entanglement of our walk as a function of coin operators for local

and non-local initial conditions is obtained. Conclusions are given in the last part, section 5.

2 Definition of the quantum walk

The discrete-time quantum walk (DTQW) on a line is defined on a Hilbert space $\mathbf{H} = \mathbf{H_p} \otimes \mathbf{H_c}$, where $\mathbf{H_c}$ is the coin Hilbert space, spanned by the basis state $|L\rangle = [1, 0]^T$ and $|R\rangle = [0, 1]^T$, where T denotes the transposed operator, that represents two sides of the directions of the motion and $\mathbf{H_p}$ is the position Hilbert space, spanned by the infinite basis states $\{|x\rangle; x \in \mathbf{Z}\}$ that represent the position of the walker. In one step of the QW we first make superposition on the coin space with a coin operator $C \in U(2)$ and after that we move the particle according to the coin state with the translation operator S as $U = S.(I_p \otimes C)$, where S is defined as $S = \Sigma_x |x-1\rangle\langle x| \otimes |L\rangle\langle L| + |x+1\rangle\langle x| \otimes |R\rangle\langle R|$, and I_p is the identity operator in $\mathbf{H_p}$. The most widely studied form of the DTQW is the walk using the Hadamard operation as quantum coin. It is an unbiased coin operation, the resulting walk is known as the Hadamard walk, which determined by the Hadamard gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
The state of QW at time t is expressed by

$$|\Psi(t)\rangle = \sum_x |x\rangle \otimes |\psi(x,t)\rangle, \tag{2-1}$$

where $|\psi(x,t)\rangle = a(x,t)|L\rangle + b(x,t)|R\rangle$ denotes the coin state and $a(x,t)$ ($b(x,t)$) is the amplitude of the base $|x, L\rangle$ ($|x, R\rangle$) at time t which belong to the complex number $\mathbf{C}$, satisfying the normalization condition $\sum_x |a(x,t)|^2 + |b(x,t)|^2 = 1$. Total state after one steps, is given by $|\Psi(t+1)\rangle = U|\Psi(t)\rangle$. The Fourier transform, as first noted in this context by Nayak and Vishwanath [4], is extremely useful when single-step displacements are involved because the evolution operator is diagonal in k-space. The Fourier transform $|\tilde{\psi}(k,t)\rangle$ ($k \in (-\pi, \pi)$) is given by

$$|\tilde{\psi}(k,t)\rangle = \sum_x e^{-ikx} |\psi(x,t)\rangle, \tag{2-2}$$

and by the inverse Fourier transform, we have

$$|\psi(x,t)\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx} |\tilde{\psi}(k,t)\rangle. \tag{2-3}$$

The time evolution of $|\tilde{\psi}(k,t)\rangle$ after one step of the walk is $|\tilde{\psi}(k,t+1)\rangle = \tilde{U}|\tilde{\psi}(k,t)\rangle$, where $\tilde{U} = R(k)U$ and $R(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$.

3 Entropy of entanglement

As mentioned above, the conditional shift in evolution operator of a DTQW entangles the coin and position degrees of freedom of the walker. To efficiently make use of entanglement as a physical resource, the amount of entanglement in a given system has to be quantified. Entanglement in a pure bipartite system is quantified using standard measures known as entropy of entanglement corresponds to the von Neumann entropy [51]. It is a functional of the eigenvalues of the reduced density matrix and is given by the formula:

$$S_E = -tr(\rho_c \log_2 \rho_c). \tag{3-4}$$

In this equation, $\rho_c = tr_p(\rho)$ is the reduced density operator obtained from $\rho = (U)^t \rho(t = 0)(U^\dagger)^t$ by tracing out the position degrees of freedom. Since ρ_c has two dimension, this quantity is $S_E \in [0,1]$, i.e., $S_E = 0$ for a product state and $S_E = 1$ for a maximally entangled state. Note that, in general $tr(\rho_c) = 1$ and $tr(\rho_c^2) \leq 1$. The entropy of entanglement can be obtained after digitalization of ρ_c. This operator which acts in H_c is represented by the Hermitian matrix as

$$\rho_c = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta^*(t) & \gamma(t) \end{pmatrix}, \tag{3-5}$$

where

$$\alpha(t) \equiv \sum_x |a(x,t)|^2 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{a}(k,t)|^2,$$

$$\beta(t) \equiv \sum_x a(x,t) b^*(x,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \tilde{a}(k,t) \tilde{b}^*(k,t),$$

$$\gamma(t) \equiv \sum_x |b(x,t)|^2 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{b}(k,t)|^2.$$

Where $*$ denotes the conjugate value. The eigenvalues $r_{1,2}$ of ρ_c, can be computed as

$$r_{1,2} = \frac{1}{2}[1 \pm \sqrt{1 + 4(|\beta(t)|^2 - \alpha(t)\gamma(t))}]. \tag{3-6}$$

Therefore, by using Eq. (3-4) one can obtain the entropy of entanglement as

$$S_E = -(r_1 \log r_1 + r_2 \log r_2). \tag{3-7}$$

To study asymptotic entanglement for Hadamard walk, this method has been also used by Abal et al. [43]. They show analytically that for localized particle with every initial coin states, the asymptotic entanglement is 0.872 and when nonlocal initial conditions $|\Psi_{\pm}(0)\rangle = \frac{1}{2}(|-1\rangle \pm |1\rangle) \otimes (|L\rangle + i|R\rangle)$ are considered, the asymptotic entanglement varies smoothly between almost complete entanglement, $S_E^+ \approx 0.979$, and no entanglement (product state), $S_E^- \approx 0.661$.

4 Two-period QWs

When one quantum coin acts for all the walker position, we face with a homogeneous walk which the amplitude for moving does not depend on position but inhomogeneous walk differ from it, in that we allow the coin operator to depend on the walker position. Now we will define an inhomogeneous quantum walk in a similar manner to the standard quantum walk, which the coin operator C to be dependent on x. In the case of walks on the line, C_x could be an arbitrary unitary operator on the two-dimensional coin space. Indeed, there is no need to restrict to walks in which only moves to the nearest neighbours, more generally one could allow there to be transitions from a given point to any other point on the line. However for the purposes of most of our discussion of walks on the line we will focus on the simplest case of a transitions to nearest neighbors. Let $\{C_x; x \in \mathbf{Z}\}$ be a sequence of orthogonal matrices

with $C_{2s} = H_0$ and $C_{2s+1} = H_1$ ($s \in \mathbf{Z}$), where

$$H_\gamma = \begin{pmatrix} \cos\theta_\gamma & \sin\theta_\gamma \\ \sin\theta_\gamma & -\cos\theta_\gamma \end{pmatrix}, \tag{4-8}$$

which $\gamma = 0, 1$; $\theta_0 = 2\pi\zeta(2s), \theta_1 = 2\pi\zeta(2s+1)$ are free parameters of coin operators which can be altered to choose the quantum coin operation and $\zeta \in R$ is the inverse period of the coin operations. With this selection of coin operators, we face with a two-period quantum walks which are determined by two orthogonal matrices, and are also self-dual under the Aubry-André duality. The inhomogeneous quantum walk is restricted to the finite interval $[-Q, Q]$, when $\zeta = \frac{P}{4Q}$ with relatively prime P (odd integer) and Q. So, the limit distribution of the this inhomogeneous quantum walk, divided by any power of the time variable is localized at the origin [55]. Limit probability distribution of this two-period QWs computed by Machida et al. [50]. In the rest of this work, we shall be concerned with clarifying the asymptotic value of S_E for both local and nonlocal initial conditions.

4.1 Asymptotic entanglement from local initial conditions

Let us first consider in detail the simple case of an initial state localized at the origin with no CPE as $|\Psi(0)\rangle = |0\rangle \otimes |\psi(0,0)\rangle$, where $|\psi(0,0)\rangle = a(0,0)|L\rangle + b(0,0)|R\rangle$ is the coin state. Below, we provide an analytical explanation for the observed values of asymptotic entanglement in the local case. The time evolution for this initial state in k-space regarded as

$$\begin{cases} |\tilde{\psi}(k, 2t)\rangle = (\tilde{H}_1\tilde{H}_0)^t|\tilde{\psi}(k,0)\rangle & \text{for even times} \\ |\tilde{\psi}(k, 2t+1)\rangle = \tilde{H}_0(\tilde{H}_1\tilde{H}_0)^t|\tilde{\psi}(k,0)\rangle & \text{for odd times.} \end{cases} \tag{4-9}$$

We note that $|\tilde{\psi}(k,0)\rangle = |\psi(0,0)\rangle$ and $\tilde{H}_\gamma = R(k)H_\gamma$. Two eigenvalues of $\tilde{H}_1\tilde{H}_0$ are given by

$$\lambda_\gamma(k) = c_0c_1\cos 2k + s_0s_1 + (-1)^\gamma i\sqrt{1 - (c_0c_1\cos 2k + s_0s_1)^2}\ for\ (\gamma = 0, 1), \tag{4-10}$$

where $c_\gamma = \cos\theta_\gamma$ and $s_\gamma = \sin\theta_\gamma$. The eigenvectors $|V_\gamma(k)\rangle$ corresponding to $\lambda_\gamma(k)$ are

$$|V_\gamma(k)\rangle = \frac{1}{\sqrt{N_\gamma}} \begin{pmatrix} u(k) \\ v(k) + (-1)^\gamma w(k) \end{pmatrix}, \tag{4-11}$$

which the elements of this matrix are as follows

$$u(k) = s_0 c_1 e^{2ik} - c_0 s_1$$

$$v(k) = -ic_0 c_1 \sin 2k$$

$$w(k) = i\sqrt{1 - (c_0 c_1 \cos 2k + s_0 s_1)^2},$$

and N_γ is the normalization constant. Using the spectral decomposition for $\tilde{H}_1 \tilde{H}_0$, the Fourier transform $|\widetilde{\psi}(k, 2t)\rangle$ is expressed as

$$|\widetilde{\psi}(k, 2t)\rangle = \sum_{\gamma=0}^{1} \lambda_\gamma^t(k) \langle V_\gamma(k)|\widetilde{\psi}(k, 0)\rangle |V_\gamma(k)\rangle, \tag{4-12}$$

and its spinor components are obtain

$$\begin{aligned} \widetilde{a}(k, 2t) &= u\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) + \tfrac{\lambda_1^t(k)}{N_1} G(k)\right), \\ \widetilde{b}(k, 2t) &= v(k)\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) + \tfrac{\lambda_1^t(k)}{N_1} G(k)\right) + w(k)\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) - \tfrac{\lambda_1^t(k)}{N_1} G(k)\right), \end{aligned} \tag{4-13}$$

where

$$F(k) = u^*(k)\tilde{a}(k, 0) + (v^*(k) + w^*(k))\tilde{b}(k, 0),$$

$$G(k) = u^*(k)\tilde{a}(k, 0) + (v^*(k) - w^*(k))\tilde{b}(k, 0).$$

Quantum walk after $2t + 1$ steps is

$$|\widetilde{\psi}(k, 2t + 1)\rangle = \sum_{\gamma=0}^{1} \lambda_\gamma^t(k) \langle V_\gamma(k)|\widetilde{\psi}(k, 0)\rangle \tilde{H}_0 |V_\gamma(k)\rangle. \tag{4-14}$$

It is easy to verify that coefficients of $|L\rangle$ and $|R\rangle$ have the forms as

$$\begin{aligned} \widetilde{a}(k, 2t + 1) &= e^{ik}\{(c_0 u(k) + s_0 v(k))\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) + \tfrac{\lambda_1^t(k)}{N_1} G(k)\right) + s_0 w(k)\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) - \tfrac{\lambda_1^t(k)}{N_1} G(k)\right)\} \\ \widetilde{b}(k, 2t + 1) &= e^{-ik}\{(s_0 u(k) - c_0 v(k))\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) + \tfrac{\lambda_1^t(k)}{N_1} G(k)\right) - c_0 w(k)\left(\tfrac{\lambda_0^t(k)}{N_0} F(k) - \tfrac{\lambda_1^t(k)}{N_1} G(k)\right)\}. \end{aligned} \tag{4-15}$$

The relevant quantities for the entropy entanglement are $\alpha(t)$ and $\beta(t)$ which defined in Eq. (3-5). , after $2t$ steps we obtain

$$
\begin{gathered}
\alpha(2t)=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\{|u(k)|^2(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2}) \\
+\frac{|u(k)|^2}{N_0N_1}(\lambda_0^{2t}(k)F(k)G^*(k)+\lambda_1^{2t}(k)F^*(k)G(k))\}, \\
\beta(2t)=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\{u(k)v^*(k)(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2})+u(k)w^*(k)(\frac{|F(k)|^2}{N_0^2}-\frac{|G(k)|^2}{N_1^2}) \\
+\frac{u(k)v^*(k)}{N_0N_1}(\lambda_0^{2t}(k)F(k)G^*(k)+\lambda_1^{2t}(k)F^*(k)G(k))-\frac{u(k)w^*(k)}{N_0N_1}(\lambda_0^{2t}(k)F(k)G^*(k)-\lambda_1^{2t}(k)F^*(k)G(k))\}.
\end{gathered}
\tag{4-16}
$$

The time dependence of these expressions vanishes in the long time limit by using Riemann-Lebesgue lemma (see Appendix), and the asymptotic values for even time steps are

$$
\begin{gathered}
\overline{\alpha}=\int_{-\pi}^{\pi}\frac{dk}{2\pi}|u(k)|^2(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2}), \\
\overline{\beta}=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\{u(k)v^*(k)(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2})+u(k)w^*(k)(\frac{|F(k)|^2}{N_0^2}-\frac{|G(k)|^2}{N_1^2})\}.
\end{gathered}
\tag{4-17}
$$

Where we use overline to indicate that the asymptotic limit has been taken, i.e. $\overline{\rho}_c=\lim_{t\to\infty}\rho_c$. After some algebra and use the Riemann-Lebesgue lemma the asymptotic values $\overline{\alpha}$ and $\overline{\beta}$ for odd time steps can be obtained as

$$
\begin{gathered}
\overline{\alpha}=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\{(c_0^2|u(k)|^2+c_0s_0(u(k)v^*(k)+u^*(k)v(k))+s_0^2(|v(k)|^2+|w(k)|^2))(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2}) \\
+(c_0s_0(u(k)w^*(k)+u^*(k)w(k))-2s_0^2v(k)w(k))(\frac{|F(k)|^2}{N_0^2}-\frac{|G(k)|^2}{N_1^2})\}, \\
\overline{\beta}=\int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{2ik}\{(c_0s_0(|u(k)|^2-|v(k)|^2-|w(k)|^2)-c_0^2u(k)v^*(k)+s_0^2u^*(k)v(k))(\frac{|F(k)|^2}{N_0^2}+\frac{|G(k)|^2}{N_1^2}) \\
+(-c_0^2u(k)w^*(k)+s_0^2u^*(k)w(k)+2c_0s_0v(k)w(k))(\frac{|F(k)|^2}{N_0^2}-\frac{|G(k)|^2}{N_1^2})\}.
\end{gathered}
\tag{4-18}
$$

In order to further illustrate the effects of bias coin on the asymptotic entanglement level, the asymptotic entanglement corresponding to initial condition given by $|\Psi(0)\rangle=|0\rangle\otimes\frac{|L\rangle+i|R\rangle}{\sqrt{2}}$ is shown in Fig.1 for even times and Fig.2 for odd times, discretely. The plots in these figures show the entropy of entanglement as a function of θ_0 and θ_1 defined in Eq. (3-7). As it is tangible from the form of coin operator, the period of asymptotic entanglement variations is π and the mirroring about the vertical axis indicate that $S_E(\theta_0,\theta_1)$ is an even function. Fig.1 (A and B) shows that, at the exact value $\theta_1=\pi/2$ and all possible values of θ_0 since $\bar{\alpha}=\frac{1}{2}$ and $\bar{\beta}=\frac{-i}{2}$ so the eigenvalues of ρ_c are $r_1=1$ and $r_2=0$, thus we face with a completely separable

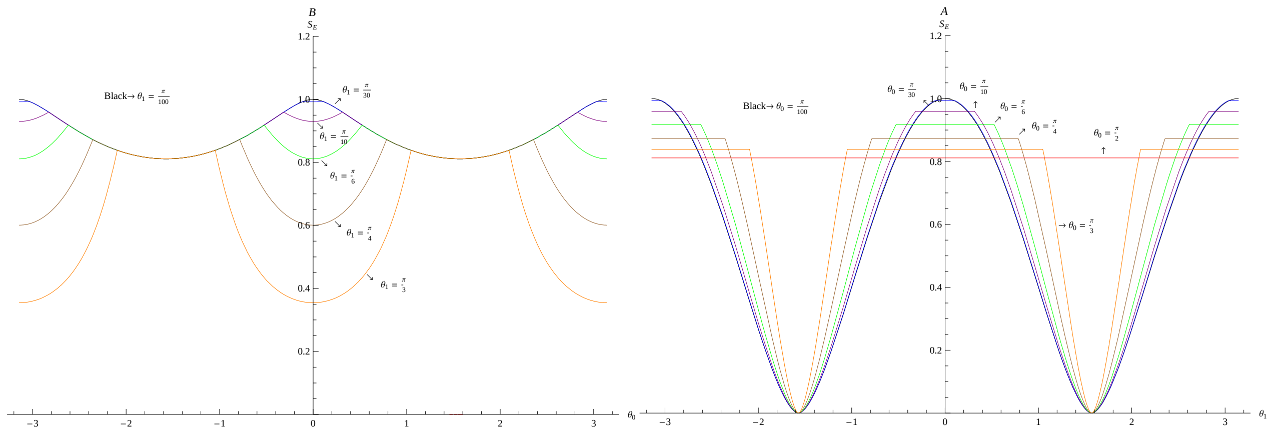

Figure 1: (Colour online) Asymptotic of entanglement S_E as a function of θ_0 and θ_1 for $\tilde{a}_0(k) = i\tilde{b}_0(k) = \frac{1}{\sqrt{2}}$ after even time steps.

state and entanglement drops to zero, $S_E^{(after\ even\ time\ steps)}(\theta_0, \frac{\pi}{2}) = 0$. Moreover for $\theta_0 = \pi/2$ and full of θ_1 we obtain $\bar{\alpha} = \frac{1}{2}$ and $\bar{\beta} = \frac{-i}{4}$ thereupon $r_1 = \frac{3}{4}$ and $r_2 = \frac{1}{4}$, in this case we have $S_E^{(after\ even\ time\ steps)}(\frac{\pi}{2}, \theta_1) = 0.811278$. In comparison, we find that, $S_E^{(after\ odd\ time\ steps)}$ versus θ_0 and θ_1 is symmetric and when these variables take the accurate quantity $\frac{\pi}{2}$, we have $\bar{\alpha} = \frac{1}{2}$ and $\bar{\beta} = 0$ therefore maximal entanglement shows itself, see Fig.2 (A and B). To elaborate this further, we will consider the steps of evolution of the DTQW using quantum coin operations given by $H_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{pmatrix}$. After several steps of the DTQW, the state can be written as

$$\begin{cases} |\psi(2t)\rangle = \frac{(i)^t}{2\sqrt{2}}\{(e^{-it\theta_1} + (-1)^t e^{it\theta_1})|0\rangle \otimes (|L\rangle + i|R\rangle) \\ \quad +(e^{-it\theta_1} - (-1)^t e^{it\theta_1})(|-2\rangle \otimes |L\rangle + i|2\rangle \otimes |R\rangle)\}, \\ |\psi(2t+1)\rangle = \frac{(i)^t}{2\sqrt{2}}\{(e^{-it\theta_1} + (-1)^t e^{it\theta_1})(i|-1\rangle \otimes |L\rangle + |1\rangle \otimes |R\rangle) \\ \quad +(e^{-it\theta_1} - (-1)^t e^{it\theta_1})(i|1\rangle \otimes |L\rangle + |-1\rangle \otimes |R\rangle)\}. \end{cases} \tag{4-19}$$

We can observe that, when H_0 reduces to the Pauli X operator, then the inverse period of the coin operations corresponds to $\frac{1}{4\times 2}$, so our inhomogeneous quantum random walk is restricted

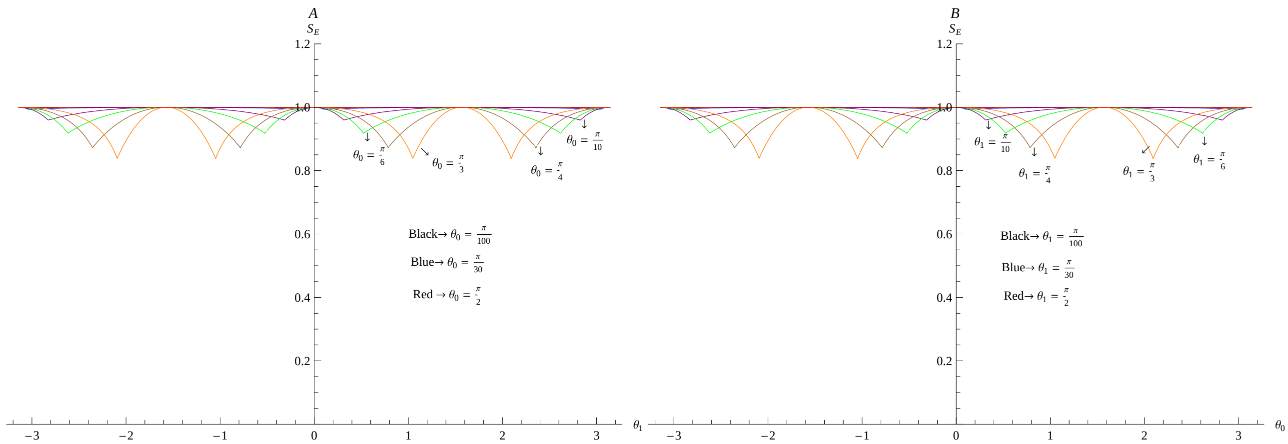

Figure 2: (Colour online) Asymptotic of entanglement S_E as a function of θ_0 and θ_1 for $\tilde{a}_0(k) = i\tilde{b}_0(k) = \frac{1}{\sqrt{2}}$ after odd time steps.

to the finite interval $[-2, 2]$ [55]. The reduced density operator can be computed from above equation:

$$\rho_c(2t) = \begin{pmatrix} \frac{1}{2} & \frac{-i}{4}(1 + (-1)^t \cos(2t\theta_1)) \\ \frac{i}{4}(1 + (-1)^t \cos(2t\theta_1)) & \frac{1}{2} \end{pmatrix} \quad and \quad \rho_c(2t+1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

after averaging over t in the long time, the matrix element of the reduced density operator is obtained:

$$\overline{\rho}_c(after\ enen\ steps) = \begin{pmatrix} \frac{1}{2} & \frac{-i}{4} \\ \frac{i}{4} & \frac{1}{2} \end{pmatrix} \quad and \quad \overline{\rho}_c(after\ odd\ steps) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that $S_E^{(after\ even\ time\ steps)}(\frac{\pi}{2}, \theta_1) = 0.811278$ and $S_E^{(after\ odd\ time\ steps)}(\frac{\pi}{2}, \theta_1) = 1$, see Figs.1 and 2. When we let H_1 diminish to coin shift operator and other parameter of coin (θ_0) alter, the wave functions of the QW can be written as

$$\begin{cases} |\psi(2t)\rangle = \frac{(-i)^t}{\sqrt{2}} e^{it\theta_0} |0\rangle \otimes (|L\rangle + i|R\rangle) \\ |\psi(2t+1)\rangle = \frac{(-i)^t}{\sqrt{2}} e^{i(t+1)\theta_0} (|-1\rangle \otimes |L\rangle - i|1\rangle \otimes |R\rangle). \end{cases} \tag{4-20}$$

It is discernible that particle remain in the origin in all even times as a separable state and in all odd times, it with maximal entanglement exist in the positions -1 and $+1$ with coin states $|L\rangle$ and $|R\rangle$, respectively. Hence, the quantum walk is bounded.

One of the most remarkable facets of this walk, is behavior of asymptotic CPE for $-\theta_0 \leq \theta_1 \leq \theta_0$ in all even times. In this interval $S_E^{(after\ even\ time\ steps)}$ gets stuck value which is determine by biases of H_0, this characteristic ascertain that when θ_0 attains constant value, conversions of θ_1 can not change the level of entanglement. The change of asymptotic CPE is from $\theta_1 \in (\theta_0, \pi - \theta_0)$, in this interval $S_E^{(after\ even\ time\ steps)}$ fluctuate between the maximum and minimum possible entanglement levels and its quantity depend on both parameters of coin operator, θ_0 and θ_1 (see $Fig.$ 1 (A)). Moreover, with increase the value of θ_0 from zero to $\frac{\pi}{2}$, maximum quantity of long time limit entanglement reduce. For example we study the behavior of asymptotic entanglement after even time steps for $\theta_0 = \frac{\pi}{4}$ and we find that $\overline{\alpha} = \frac{1}{2}$ and for $\theta_1 = \frac{\pm\pi}{4}, \frac{\pm 3\pi}{4}$, $\overline{\beta} = \frac{i}{2}(1-\sqrt{2})$ but against other values of θ_1 exclusion of $\frac{\pm\pi}{2}$ we have

$$\begin{gathered}
\overline{\beta} = \frac{1}{8\pi} \sec^3\theta_1 \{3i\pi\cos\theta_1 + i\pi\cos 3\theta_1 + 2\cos^{5/2} \\
(\sqrt{3\cos\theta_1 - \cos 3\theta_1 + 4\sqrt{1+\cos 2\theta_1}\sin\theta_1}\{\ln[\frac{i(1+\cos 2\theta_1 - \sin 2\theta_1 - 2\sqrt{1+\cos 2\theta_1})}{\sqrt{6\cos^2\theta_1 - 2\cos\theta_1\cos 3\theta_1 + 4\sin 2\theta_1\sqrt{1+\cos 2\theta_1}}}] \\
- \ln[\frac{-i(1+\cos 2\theta_1 - \sin 2\theta_1 - 2\sqrt{1+\cos 2\theta_1})}{\sqrt{6\cos^2\theta_1 - 2\cos\theta_1\cos 3\theta_1 + 4\sin 2\theta_1\sqrt{1+\cos 2\theta_1}}}]\} \\
+ i\sqrt{-3\cos\theta_1 + \cos 3\theta_1 + 4\sqrt{1+\cos 2\theta_1}\sin\theta_1}\{\ln[\frac{-\sqrt{2}(1+\cos 2\theta_1 - \sin 2\theta_1 + 2\sqrt{1+\cos 2\theta_1})}{\sqrt{-3\cos^2\theta_1 + \cos\theta_1\cos 3\theta_1 + 2\sin 2\theta_1\sqrt{1+\cos 2\theta_1}}}] \\
- \ln[\frac{\sqrt{2}(1+\cos 2\theta_1 - \sin 2\theta_1 + 2\sqrt{1+\cos 2\theta_1})}{\sqrt{-3\cos^2\theta_1 + \cos\theta_1\cos 3\theta_1 + 2\sin 2\theta_1\sqrt{1+\cos 2\theta_1}}}]\})\}.
\end{gathered} \tag{4-21}$$

when $0 \leq \theta_1 \leq \frac{\pi}{4}$, the exact value of asymptotic CPE is $S_E^{(after\ even\ time\ steps)}(\frac{\pi}{4}, \theta_1 \leq \frac{\pi}{4}) = 0.872429$, this precise value are coincident with that obtained for homogeneous Hadamard walk with local initial condition [43], from $\frac{\pi}{4}$ to $\pi - \frac{\pi}{4}$ asymptotic CPE variate slowly and achieve zero for correct $\theta_1 = \frac{\pm\pi}{2}$. While θ_1 close to $\pi - \frac{\pi}{4}$, it gradually augment to 0.872429 when the parameter θ_1 attains $\pi - \frac{\pi}{4}$ and the asymptotic entanglement maintain this value to $\theta_1 = \pi$. In plot (B) presented in $Fig.$ 1, overlapping graphs endorse our notion that for $\theta_1 \leq \theta_0$ the limiting value of the entanglement is regulated with coin operator H_0. For example after

even time steps we obtain $\overline{\alpha} = \frac{1}{2}$ and for $\theta_0 = \frac{\pm\pi}{4}, \frac{\pm 3\pi}{4}$, $\overline{\beta} = \frac{i}{2}(1-\sqrt{2})$ but versus other values of θ_0 exclusion of $\frac{\pm\pi}{2}$ we have

$$\begin{aligned}
\overline{\beta} = & \frac{1}{4\pi\sqrt{1+\cos 2\theta_0}\sqrt{-3\cos\theta_0+\cos 3\theta_0+4\sin\theta_0\sqrt{1+\cos 2\theta_0}}}\{i\sec^{3/2}\theta_0(-3\cos\theta_0+i\cos 3\theta_0+4\sin\theta_0\sqrt{1+\cos 2\theta_0}) \\
& \{\ln[\frac{-(1+\cos 2\theta_0-\sin 2\theta_0+2\sqrt{1+\cos 2\theta_0})}{\sqrt{-3\cos^2\theta_0+\cos\theta_0\cos 3\theta_0+2\sin 2\theta_0\sqrt{1+\cos 2\theta_0}}}] - \ln[\frac{(1+\cos 2\theta_0-\sin 2\theta_0+2\sqrt{1+\cos 2\theta_0})}{\sqrt{-3\cos^2\theta_0+\cos\theta_0\cos 3\theta_0+2\sin 2\theta_0\sqrt{1+\cos 2\theta_0}}}]\} \\
& +i\sec^{3/2}\theta_0\sqrt{2\sin^2 2\theta_0-(3\cos\theta_0-\cos 3\theta_0)^2} \\
& \{\ln[\frac{i(1+\cos 2\theta_0-\sin 2\theta_0-2\sqrt{1+\cos 2\theta_0})}{\sqrt{3\cos^2\theta_0-\cos\theta_0\cos 3\theta_0+2\sin 2\theta_0\sqrt{1+\cos 2\theta_0}}}] - \ln[\frac{-i(1+\cos 2\theta_0-\sin 2\theta_0-2\sqrt{1+\cos 2\theta_0})}{\sqrt{3\cos^2\theta_0-\cos\theta_0\cos 3\theta_0+2\sin 2\theta_0\sqrt{1+\cos 2\theta_0}}}]\} \\
& +2\sqrt{2}i\sqrt{1+\cos 2\theta_0}\sqrt{-3\cos\theta_0+\cos 3\theta_0+4\sin\theta_0\sqrt{1+\cos 2\theta_1}}\tan^2\theta_0\}.
\end{aligned} \tag{4-22}$$

4.2 Asymptotic entanglement from nonlocal initial conditions

Asymptotic entanglement levels for nonlocal initial conditions are reported for the first time in the context of the homogeneous Hadamard walk on the line. In order to show that the asymptotic entanglement level is strongly dependent on whether the initial condition is localized or delocalized in position space. In this section we apply our inhomogeneous walk to consider in detail the case of initial conditions in the position subspace $\mathbf{H_p}$ spanned by $|\pm 1\rangle$, as

$$|\Psi(0)\rangle = \frac{|-1\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|L\rangle + i|R\rangle}{\sqrt{2}}. \tag{4-23}$$

Whit the projection on k-space this non localized state is simply $|\tilde{\psi}(k,0)\rangle = \cos(k)(|L\rangle + i|R\rangle)$. The dependence on the initial conditions is contained in the coin operator. The time evolution of the motion in phase space for the initial states defined in the previous equation can be expressed by

$$\begin{cases} |\tilde{\psi}(k,2t)\rangle = (\tilde{H}_0\tilde{H}_1)^t|\tilde{\psi}(k,0)\rangle & \text{for even times} \\ |\tilde{\psi}(k,2t+1)\rangle = \tilde{H}_1(\tilde{H}_0\tilde{H}_1)^t|\tilde{\psi}(k,0)\rangle & \text{for odd times.} \end{cases} \tag{4-24}$$

One can check that the eigenvalues of $\tilde{H}_0\tilde{H}_1$ are equivalent with Eq. (4-10), exactly, and the normalized eigenvectors are of the form Eq. (4-11) with components

$$u'(k) = s_1 c_0 e^{2ik} - c_1 s_0$$

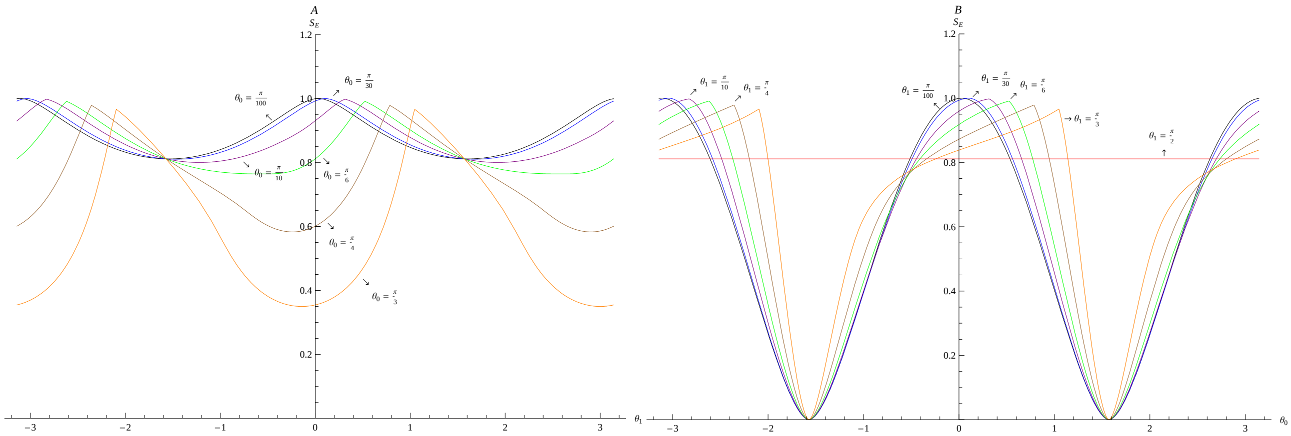

Figure 3: (Colour online) Asymptotic of entanglement S_E as a function of θ_0 and θ_1 for $\tilde{a}_0(k) = i\tilde{b}_0(k) = \cos(k)$ in even times.

$$v(k) = -ic_0c_1 \sin 2k$$

$$w(k) = i\sqrt{1-(c_0c_1\cos 2k + s_0s_1)^2}.$$

As in previous subsection relate in details, the matrix elements of the reduced density operator after even times is obtained as

$$\begin{gathered}\overline{\alpha} = \int_{-\pi}^{\pi}\tfrac{dk}{2\pi}|u'(k)|^2(\tfrac{|F(k)|^2}{N_0^2}+\tfrac{|G(k)|^2}{N_1^2}),\\ \overline{\beta} = \int_{-\pi}^{\pi}\tfrac{dk}{2\pi}\{u'(k)v^*(k)(\tfrac{|F(k)|^2}{N_0^2}+\tfrac{|G(k)|^2}{N_1^2})+u'(k)w^*(k)(\tfrac{|F(k)|^2}{N_0^2}-\tfrac{|G(k)|^2}{N_1^2})\}.\end{gathered} \tag{4-25}$$

Subsequently, after odd times the relevant quantities for the entropy of entanglement are

$$\begin{gathered}\overline{\alpha} = \int_{-\pi}^{\pi}\tfrac{dk}{2\pi}\{(c_1^2|u'(k)|^2 + c_1s_1(u'(k)v^*(k)+u'^*(k)v(k)) + s_1^2(|v(k)|^2+|w(k)|^2))(\tfrac{|F(k)|^2}{N_0^2}+\tfrac{|G(k)|^2}{N_1^2})\\ +(c_1s_1(u'(k)w^*(k)+u'^*(k)w(k)) - 2s_1^2v(k)w(k))(\tfrac{|F(k)|^2}{N_0^2}-\tfrac{|G(k)|^2}{N_1^2})\},\\ \overline{\beta} = \int_{-\pi}^{\pi}\tfrac{dk}{2\pi}e^{2ik}\{(c_1s_1(|u'(k)|^2-|v(k)|^2-|w(k)|^2) - c_1^2u'(k)v^*(k) + s_1^2u'^*(k)v(k))(\tfrac{|F(k)|^2}{N_0^2}+\tfrac{|G(k)|^2}{N_1^2})\\ +(-c_1^2u'(k)w^*(k)+s_1^2u'^*(k)w(k)+2c_1s_1v(k)w(k))(\tfrac{|F(k)|^2}{N_0^2}-\tfrac{|G(k)|^2}{N_1^2})\}.\end{gathered} \tag{4-26}$$

From these two above equations, one readily obtains the independent elements of $\overline{\rho}_c$ for this kind of initial conditions. *Fig.* 3 (*A and B*) shows the entropy of entanglement as a function of θ_0 and θ_1 after even time steps and *Fig.* 4 (*A and B*) shows it after odd time

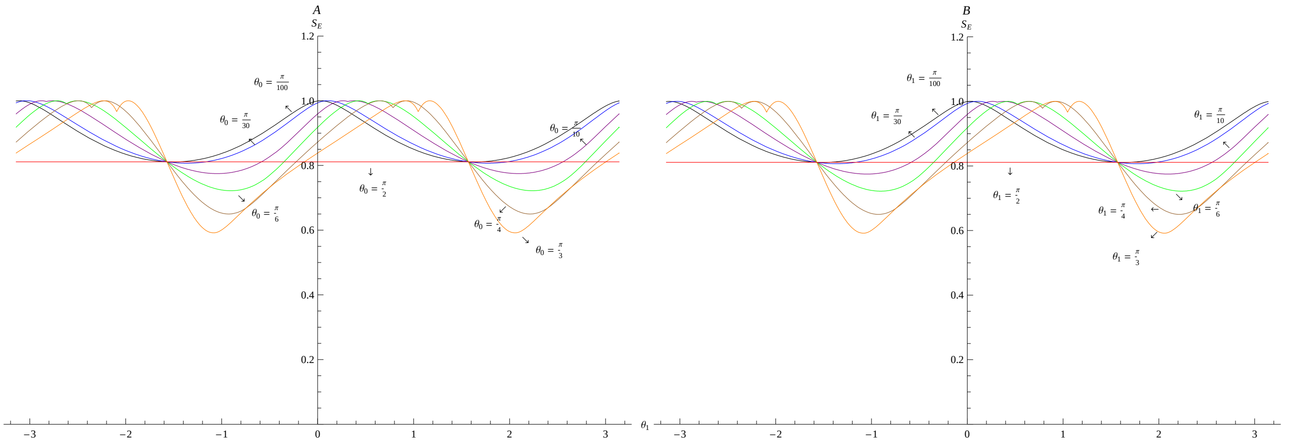

Figure 4: (Colour online) Asymptotic of entanglement S_E as a function of θ_0 and θ_1 for $\tilde{a}_0(k) = i\tilde{b}_0(k) = \cos(k)$ in odd times.

steps. In comparison with local initial condition, the first aspect of these graphs to highlight is that, asymptotic entanglement in odd times versus both variations of coin operator is symmetric and this distinguishing quality is free from type of initial conditions. The second aspect is that, $S_E^{(after\ even\ time\ steps)}$ vanishes which θ_0 tends to become $\frac{\pi}{2}$, also there is $S_E^{(after\ even\ time\ steps)}(\theta_0, \frac{\pi}{2}) = 0.811278$, (see $Fig.$ 3). The origin of all the above facts can be clarified by using the analytical expressions. Let us begin by $H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H_0 = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & -\cos\theta_0 \end{pmatrix}$, the wave functions of QW are

$$\begin{cases} |\psi(2t)\rangle = \frac{(i)^t}{4}\{(e^{-it\theta_0} + (-1)^t e^{it\theta_0})(|1\rangle \otimes |L\rangle + i|-1\rangle \otimes |R\rangle) \\ +(e^{-it\theta_0} - (-1)^t e^{it\theta_0})(|-3\rangle \otimes |L\rangle + i|3\rangle \otimes |R\rangle)\} + \frac{(i)^t}{2} e^{-it\theta_0}(|-1\rangle \otimes |L\rangle + i|1\rangle \otimes |R\rangle), \\ |\psi(2t+1)\rangle = \frac{(i)^t}{4}\{(e^{-it\theta_0} + (-1)^t e^{it\theta_0})(i|-2\rangle \otimes |L\rangle + |2\rangle \otimes |R\rangle) \\ +(e^{-it\theta_0} - (-1)^t e^{it\theta_0})(i|2\rangle \otimes |L\rangle + |-2\rangle \otimes |R\rangle)\} + \frac{(i)^t}{2} e^{-it\theta_0}|0\rangle \otimes (i|L\rangle + |R\rangle). \end{cases} \tag{4-27}$$

Density operator after some manipulation, can be expressed as

$$\rho_c(2t) = \begin{pmatrix} \frac{1}{2} & \frac{-i}{4}(1 + (i)^t e^{2it\theta_0}) \\ \frac{i}{4}(1 + (-i)^t e^{-2it\theta_0}) & \frac{1}{2} \end{pmatrix} \quad and \quad \rho_c(2t+1) = \begin{pmatrix} \frac{1}{2} & \frac{i}{4} \\ \frac{-i}{4} & \frac{1}{2} \end{pmatrix},$$

in the long-time limit, the contribution of the time dependent terms in these ρ_cs vanishes and we have

$$\overline{\rho}_c(after\ enen\ steps) = \begin{pmatrix} \frac{1}{2} & \frac{-i}{4} \\ \frac{i}{4} & \frac{1}{2} \end{pmatrix} \quad and \quad \overline{\rho}_c(after\ odd\ steps) = \begin{pmatrix} \frac{1}{2} & \frac{i}{4} \\ \frac{-i}{4} & \frac{1}{2} \end{pmatrix}.$$

From Eq. (3-6), the exact eigenvalues of the both operators are $r_1 = \frac{3}{4}$ and $r_2 = \frac{1}{4}$, these eigenvalues yield the asymptotic value for the entropy of entanglement

$$S_E = -(\frac{3}{4}\log_2(\frac{3}{4}) + \frac{1}{4}\log_2(\frac{1}{4})) = 0.811278.$$

If H_0 replace by the shift coin operator, then the position states are

$$\begin{cases} |\psi(2t)\rangle = \frac{(-i)^t}{2}e^{it\theta_1}(|-1\rangle + |1\rangle) \otimes (|L\rangle + i|R\rangle) \\ |\psi(2t+1)\rangle = \frac{(-i)^t}{\sqrt{2}}e^{i(t+1)\theta_1}|0\rangle \otimes (|L\rangle + i|R\rangle). \end{cases} \tag{4-28}$$

Walk is bounded in this cases, overtly, the reduce density operators are given by

$$\rho_c(2t) = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix} \quad and \quad \rho_c(2t+1) = \begin{pmatrix} \frac{1}{2} & \frac{i}{4} \\ \frac{-i}{4} & \frac{1}{2} \end{pmatrix},$$

we observe that entropy of entanglement is zero and 0.811278 in the even and odd times respectively.

5 Conclusion and discuss

We have investigated an inhomogeneous walk with two-period, to be used in quantum random walk in one dimension latices. It determined by two orthogonal matrices and included two free parameters that together provided many conditions under which a measurement performed on the coin state yielded the value of entanglement on the resulting position quantum state. We have studied the problem analytically for all values of two free parameters of coin with diverse initial conditions for large number of QW steps. To summarise the results for this walk, we find the various behaviours of the entanglement are governed as follows. We demonstrated

that **how** the asymptotic value depend on coin parameters and initial condition. Moreover it was different for odd or even positions and when on the coin parameters take the precise value $\frac{\pi}{2}$ walk altered to a bounded motion. Two striking characteristics of this walk was that, when motion began with local initial condition, walk regulated by the first coin operator that act on the state of particle in all even tims and $S_E^{(after\ odd\ time\ steps)}$ was symmetric ratio θ_0 and θ_1 for all type of initial conditions.

Appendix

The Riemann-Lebesgue lemma

This little note is devoted to a proof of the Riemann-Lebesgue lemma. We use the following notation for the nth Fourier coefficient of a 2π-periodic function f:

$$f(n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \tilde{f}(k) e^{ink}.$$

Lemma: Assume that $\tilde{f}(k)$ is 2π-periodic, bounded and integrable. Then $f(n) \to 0$ when $n \to \pm\infty$.

Proof: We shall prove this only for real valued functions. If $\tilde{f}(k)$ is complex valued, the result will follow from the result applied to the real and imaginary parts of $\tilde{f}(k)$ separately. First, we prove the result for an extremely special case: Namely, a single step, which is a function of the form

$$\tilde{s}(k) = \begin{cases} 1 & a + 2k\pi \leq k \leq b + 2k\pi, \quad k \in \mathbf{Z} \\ 0 & otherwise \end{cases}$$

where $a < b$ and $b - a < 2\pi$. Then

$$s(n) = \int_a^b \frac{dk}{2\pi} e^{ink} = \frac{e^{inb} - e^{ina}}{2\pi in} \to 0 \quad as\ n \to \pm\infty,$$

since the numerator is bounded and the denominator goes to infinity. Second, since any step function is a linear combination of a finite number of single steps, the same result holds for

step functions.

Finally, now assume that $\tilde{f}(k)$ is integrable, and pick any $\varepsilon > 0$. It follows - practically direct from the definition of integrability - that there exists a step function $\tilde{s}(k)$ with

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{f}(k) - \tilde{s}(k)| < \varepsilon.$$

From this we get

$$|f(n) - s(n)| = |\int_{-\pi}^{\pi} \frac{dk}{2\pi} (\tilde{f}(k) - \tilde{s}(k))e^{ink}| \leq \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{f}(k) - \tilde{s}(k)| < \varepsilon,$$

as well. We have shown that $s(n) \to 0$, so there is some N so that $|n| \geq N$ implies $s(n) < \varepsilon$. Whenever $|n| \geq N$, then

$$|f(n)| \leq |f(n) - s(n)| + |s(n)| < \varepsilon + \varepsilon = 2\varepsilon,$$

which finishes the proof.

Notice that the Riemann-Lebesgue lemma says nothing about how fast $f(n)$ goes to zero. With just a bit more of a regularity assumption on $\tilde{f}(k)$, we can show that $f(n)$ behaves roughly like $1/n$ or better. This is easy if $\tilde{f}(k)$ is continuous and piecewise smooth, as is seen from the identity $f'(n) = i\ nf(n)$, which arises from partial integration. Applying the RiemannLebesgue lemma to $\tilde{f}'(k)$ we conclude that $f(n)$ is $1/n$ times something that goes to zero, so $f(n) \to 0$ faster than $1/n$, [54].

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